The free pseudospace is n-ample, but not $(n+1)$-ample

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Abstract

We give a uniform construction of free pseudospaces of dimension n extending work in [1]. This yields examples of ω-stable theories which are n-ample, but not $n+1$-ample. The prime models of these theories are buildings associated to certain right-angled Coxeter groups.

1 Introduction

In the investigation of geometries on strongly minimal sets the notion of ampleness plays an important role. Algebraically closed fields are n-ample for all n and it is not known whether there are strongly minimal sets which are n-ample for all n and do not interpret an infinite field. Obviously, one way of proving that no infinite field is interpretable in a theory is by showing that the theory is *not* n-ample for some n.

In [1], Baudisch and Pillay constructed a free pseudospace of dimension 2. Its theory is ω-stable (of infinite rank) and 2-ample. F. Wagner posed the question whether this example was 3-ample or not.

In Section 2 we give a uniform construction of a free pseudospace of dimension n and show that it is n-ample, but not $n+1$-ample. It turns out that the theory of the free pseudospace of dimension n is the first order theory of a Tits-building associated to a certain Coxeter diagram and we will investigate this connection in Section 4.

In the final section we show that there are exactly two orthogonality classes of regular types.

2 Construction and results

Fix a natural number $n \geq 1$. Let L_n be the language for $n+1$-coloured graphs containing predicates $V_i, i = 0, \ldots n$ and an edge relation E.

By an L_n-graph we mean an $n+1$-coloured graph with vertices of types $V_i, i = 0, \ldots n$ and an edge relation $E \subseteq \bigcup_{i=1,\ldots n} V_{i-1} \times V_i$. We say that a path in this graph is of type E_i if all its vertices are in $V_{i-1} \cup V_i$ and of type $E_i \cup \ldots \cup E_{i+j}$ if all its vertices are in $V_{i-1} \cup \ldots \cup V_{i+j}$

The free pseudospaces will be modeled along the lines of a projective space, i.e. we will think of vertices of type V_i as i-dimensional spaces in a free pseudospace. Therefore we extend the notion of incidence as follows:

Definition 2.1. *1. We say that a vertex x_i of type V_i is* incident *to a vertex x_j of type V_j if there are vertice x_l of type $V_l, l = i+1 \ldots j$ such that $E(x_{l-1}, x_l)$ holds. In this case the sequence $(x_i, \ldots x_j)$ is called a* dense flag. *A* flag *is a sequence of vertices $(x_1, \ldots x_k)$ in which any two vertices are incident.*

2. *The* residue $R(x)$ *of a vertex x is the set of vertices incident with x.*

3. *We say that two vertices x and y* intersect in the vertex z *and write $z = x \wedge y$ if the set of vertices of type V_0 incident with x and y is exactly the set of vertices of type V_0 incident with z. If there is no vertex of type V_0 incident to x and y, we say that x and y intersect in the empty set.*

4. *We say that two vertices x and y* generate the vertex z *and write $z = x \vee y$, if the set of vertices of type V_n incident with x and y is exactly the set of vertices of type V_n incident with z. If there is no vertex of type V_n incident to x and y, we say that x and y generate the empty set.*

5. *A* simple cycle *is a cycle without repetitions.*

We now give an inductive definition of a free pseudospace of dimension n:

Definition 2.2. *A free pseudospace of dimension 1 is a free pseudoplane, i.e. an L_1-graph which does not contain any cycles and such that any vertex has infinitely many neighbours.*

Assume that a free pseudospace of dimension $n-1$ has been defined. Then a free pseudospace of dimension n is an L_n-graph such that the following holds:

$(\Sigma 1)_n$ *(a) The set of vertices of type $V_0 \cup \ldots \cup V_{n-1}$ is a free pseudospace of dimension $n-1$.*

(b) The set of vertices of type $V_1 \cup \ldots \cup V_n$ is a free pseudospace of dimension $(n-1)$.

$(\Sigma 2)_n$ *(a) For any vertex x of type V_0, $R(x)$ is a free pseudospace of dimension $(n-1)$.*

(b) For any vertex x of type V_n, $R(x)$ is a free pseudospace of dimension $(n-1)$.

$(\Sigma 3)_n$ *(a) Any two vertices x and y intersect in some vertex z or the emptyset.*

(b) Any two vertices x and y generate some vertex z or the emptyset.

$(\Sigma 4)_n$ *(a) If a is a vertex of type V_n and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k, then there is an $E_1 \cup \ldots \cup E_{n-1}$-path from b to b' of length at most $k-1$ in $R(a)$ all of whose V_0-vertices appear in γ.*

(b) If a is a vertex of type V_0 and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k, then there is an $E_2 \cup \ldots \cup E_n$-path from b to b' of length at most $k-1$ in $R(a)$ all of whose V_n-vertices appear in γ.

Let T_n denote the L_n-theory expressing these axioms.

Note that the inductive nature of the definition immediately has the following consequences:

1. The induced subgraph on $V_j \cup \ldots \cup V_{j+m}$ is a free pseudospace of dimension m.

2. If a is a vertex of type V_{j+m} and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k contained in $V_j \cup \ldots \cup V_{j+m}$, then there is an $E_{j+1} \cup \ldots \cup E_{j+m-2}$-path from b to b' of length at most $k-1$ in $R(a)$ all of whose V_j-vertices appear in γ.

3. The notion of a free pseudospace of dimension n is *self-dual*: if we put $W_i = V_{n-i}, i = 0, \ldots n$, then $W_0, \ldots W_n$ with the same set of edges is again a free pseudospace of dimension n.

Our first goal is to show that T_n is consistent and complete.

Definition 2.3. *Let A be a finite L_n-graph. The following extensions are called elementary strong extensions of A:*

1. *add a vertex of any type to A which is connected to at most one vertex of A of an appropriate type.*

2. *If (x, y, z) is a dense flag in A, add a vertex of the same type as y to A which is connected to both x and z.*

We write $A \leq B$ if B arises from A by finitely many elementary strong extensions.

Definition 2.4. *Let $\mathcal{K}_n$ be the class of finite L_n-graphs A such that the following holds*

1. *A does not contain any E_i-cycles for $i = 1, \ldots n$.*

2. *If $a \neq a'$ are in A, they intersect in a vertex of A or the emptyset.*

3. *If $a \neq a'$ are in A, they generate a vertex of A or the emptyset.*

4. *If (b, a, b') is a path with $a \in V_i, b, b' \in V_{i-1}$, and $\gamma = (a, b, \ldots, b', a)$ is an $E_i \cup E_{i-j}$ path of length k, then there is some $E_{i-1} \cup \ldots E_{i-j}$-path from b to b' of length at most $k - 1$ in $R(a)$ with all V_{i-j}-vertices occurring in γ.*

5. *If (b, a, b') is a path with $a \in V_i, b, b' \in V_{i+1}$, and $\gamma = (a, b, \ldots, b', a)$ is an $E_i \cup \ldots \cup E_{i+j}$ path of length k, then there is some $E_{i+1} \cup \ldots \cup E_{i+j}$-path from b to b' of length at most $k - 1$ in $R(a)$ with all V_{i+j}-vertices occurring in γ.*

Note that $\mathcal{K}_n$ is closed under finite substructures. We next show that $(\mathcal{K}, \leq)$ has the amalgamation property for strong extensions.

For any finite L_n-graphs $A \subseteq B, C$ we denote by $B \otimes_A C$ the *free amalgam* of B and C over A, i.e. the graph on $B \cup C$ containing no edges between elements of $B \setminus A$ and $C \setminus A$.

Lemma 2.5. *If $A \leq B, C$ are in $\mathcal{K}_n$, then $D := B \otimes_A C \in \mathcal{K}$ and $B, C \leq D$.*

Proof. Clearly, $B, C \leq D$. To see that $D \in \mathcal{K}_n$, note that if $B \in \mathcal{K}_n$ and B' is an elementary strong extension of B, then also $B' \in \mathcal{K}_n$. This is clear for strong extensions of type 1. For strong extensions of type 2. suppose that

(b, a, b') is a path with $a \in V_i, b, b' \in V_{i-1}$, and $\gamma = (a, b, \ldots, b', a) \subset B'$ is an $E_i \cup \ldots \cup E_{i-j}$-path of length k containing the new vertex y. Since the new vertex has exactly two neighbours y_1, y_2, this implies that the vertex is of type V_m for some $i - j \leq m \leq i$ and (y_1, y, y_2) is contained in γ. By construction of strong extensions, there is some $z \in B$ such that (y_1, z, y_2) is a path. Hence we may replace all occurrences of y in γ by z. Then γ is contained in B and we find the required path in $R(a)$ with all V_{i-j}-vertices occurring in γ. $\square$

This shows that the class $(\mathcal{K}_n, \leq)$ has a Hrushovski limit M_n, i.e. a countable L_n-structure M_n whose strong subsets are exactly the L_n-graphs in $\mathcal{K}_n$ and which is homogeneous for strong subsets: if $A, B \leq M_n$ then any isomorphism from A to B extends to an automorphism of M_n. Here we say as usual that a subset A of M_n is strong in M_n if $A \cap B \leq B$ for any finite set $B \subset M_n$.

Proposition 2.6. *The Hrushovski limit M_n is a model of T_n.*

Proof. By construction, $V_i \cup \ldots \cup V_{i+j}$ satisfies $(\Sigma 3)_j$ and $(\Sigma 4)_j$ for any i, j. In particular, M_n satisfies $(\Sigma 3)_n$ and $(\Sigma 4)_n$.

$(\Sigma 1)_n$: In order to show that M satisfies $(\Sigma 1)_n$, we first note that $V_i \cup V_{i+1}$ is a free pseudoplane for all $i = 0, \ldots n-1$. Assume inductively that $V_j \cup \ldots \cup V_{j+i}$ is a free pseudospace of dimension i. To see that $V_j \cup \ldots \cup V_{j+i+1}$ is a free pseudospace of dimension $i+1$, we need only verify $(\Sigma 2)_{i+1}$. Hence we have to show that for $a \in V_j$ the residue $R(a) \cap (V_j \cup \ldots \cup V_{j+i+1})$ is a free pseudospace of dimension i. We know by induction that $R(a) \cap (V_j \cup \ldots \cup V_{j+i})$ is a free pseudospace.

Clearly,

$$R(a) \cap (V_{j+1} \cup \ldots \cup V_{j+i+1}) = \bigcup \{R(b) \cap (V_{j+1} \cup \ldots \cup V_{j+i+1}) \colon b \in V_{j+1}, E(a, b)\}.$$

For each neighbour $b \in V_{j+1}$ of a, the set $R(b) \cap (V_{j+1} \cup \ldots \cup V_{j+i+1})$ is a free pseudospaces of dimension $i-1$ by induction. Since $(V_{j+1} \cup \ldots \cup V_{j+i+1})$ is a free pseudospace of dimension i, $(\Sigma 2)_{i+1}$ follows from the induction hypothesis. Hence $V_0 \cup \ldots \cup V_{n-1}$ and $V_1 \cup \ldots \cup V_n$ are free pseudospaces of dimension $n - 1$.

$(\Sigma 2)_n$: The proof of $(\Sigma 2)_n$ is similar. $\square$

We say that a model M of T_n is $\mathcal{K}_n$-saturated if for all finite $A \leq M$ and strong extensions C of A with $C \in \mathcal{K}_n$ there is a strong embedding of C into M fixing A elementwise. Clearly, by construction, M_n is $\mathcal{K}_n$-saturated.

Lemma 2.7. *A model M of T_n is ω-saturated if and only if M is $\mathcal{K}_n$-saturated.*

Proof. Let M be an ω-saturated model of T_n. To show that M is $\mathcal{K}_n$-saturated, let $A \leq M$ and $A \leq B \in \mathcal{K}_n$. By induction we may assume that F is an elementary strong extension of A and it is easy to see that F can be imbedded over A into M. $\square$

Corollary 2.8. *The theory T_n is complete.*

Proof. Let M be a model of T_n. In order to show that M is elementarily equivalent to M_n choose an ω-saturated $M' \equiv M$. By Lemma 2.7, M' is $\mathcal{K}_n$-saturated. Now M' and M_n are partially isomorphic and therefore elementarily equivalent. $\square$

We will see in Section 4 that T_n is the theory of the building of type $A_{\infty,n+1}$ with infinite valencies.

Definition 2.9. *Following* [1] *we call a subset A of a model M of T_n* nice *if*

1. *any E_i-path between elements of A lies entirely in A and*
2. *if $a, b \in A$ are connected by a path in M there is a path from a to b inside A.*

Remark 2.10. Note that a subset A of M_n is strong in M_n if and only if it is nice. (This follows immediately from the definition of strong extension.)

Since M_n is homogeneous for strong subsets, for any nice subset of M_n the quantifier-free type determines the type, i.e. if $\bar{a}, \bar{b}$ are nice in M_n and such that $\mathrm{qftp}(\bar{a}) = \mathrm{qftp}(\bar{b})$, then $\mathrm{tp}(\bar{a}) = \mathrm{tp}(\bar{b})$.

We now work in a very saturated model $\overline{M}$ of T_n.

Lemma 2.11. *If A is a finite set, there is a nice finite set B containing A.*

Proof. Since single vertices are nice it suffices to prove the following

Claim: *If A is nice and a arbitrary, then there is a nice finite set B containing $A \cup \{a\}$.*

Proof of Claim: Of course we may assume $a \notin A$. If there is no path from a to A, clearly $A \cup \{a\}$ is nice. Hence we may also assume that there is some path $\gamma = (a = x_0, \ldots b)$ for some $b \in A$ and $\gamma \cap A = \{b\}$. It therefore suffices to prove the claim for the case where a has a neighbour in A. If a has two neighbours $x, y \in A$ then (x, a, y) is a dense flag and $A \cup \{a\}$ is nice.

Now assume that $a \in V_i$ has a unique neighbour of type V_{i+1} in A. (The other case then follows by self-duality.) If the E_i-connected component of a does not intersect A, then again $A \cup \{a\}$ is nice. Otherwise there is some E_i-path $\gamma = (x_0 = a, \ldots x_m = b)$ in M_n with $\gamma \cap A = \{b\}$. If for some V_{i-1}-vertex x_k of γ there is an E_{i-1}-path to some $c \in A$, then the E_i-path from c to b extends $(x_k, \ldots x_m = b)$ and is entirely contained in A since A is nice. Since $\gamma \cap A = \{b\}$, no such x_k exists implying that $A \cup \gamma$ is nice. □

Let us say that γ *changes direction in* x_i if $x_i \in V_j$ and either $x_{i-1}, x_{i+1} \in V_{j-1}$ or $x_{i-1}, x_{i+1} \in V_{j-1}$ for some j. Clearly a path which doesn't change direction is a dense flag.

We will use the following easy consequence of $(\Sigma 4)_{j \leq n}$.

Lemma 2.12. *If $\gamma = (a, b, \ldots, b', a)$ is a simple cycle, then any two vertices $x, y \in \gamma \cap (V_{j-1} \cup V_j)$ are joined by an E_j-path.*

Proof. Suppose that $j, j+m$ are minimal and maximal, respectively, with $\gamma \cap V_j, V_{j+m} \neq \emptyset$. Apply $(\Sigma 4)_m$ to all vertices of $\gamma \cap V_{j+m}$ and replace any path of the form (x, y, z) with $y \in V_{j+m}$ by a path from x to z contained in $V_j \cup \ldots \cup V_{j+m-1}$. This yields a new path contained in $V_j \cup \ldots \cup V_{j+m-1}$. Next apply $(\Sigma 4)_{m-1}$ to each vertex of type V_j to obtain a path in $V_{j+1} \cup \ldots \cup V_{j+m-1}$. Alternating between replacing the top and bottom extremal peaks by a path using fewer levels, we end up with an E_k-path for some $j \leq k \leq j+m-1$ between vertices of γ. Since an E_k-path changes direction at every vertex, we can apply the same procedure in order to replace this path by an $E_{k'}$-path for any $j \leq k' \leq j+m-1$. □

Definition 2.13. *We call a path $\gamma = (x_0, \ldots x_k) \subseteq V_j \cup \ldots \cup V_{j+m}$* reduced *if γ is a flag or if the following holds:*

1. *if $m = 1$ the path γ is reduced if it does not contain any repetition.*

2. *for any simple cycle γ' containing $(x_i, \ldots, x_{i+q})$ replacing all (y, x, z) with $x \in (x_{i+1}, \ldots, x_{i+q-1}) \cap V_{j+m}$ by an E_{j+m-1}-path in $R(x)$ yields a reduced path $(x_i, \ldots, y, \ldots z, \ldots, x_{i+q}) \subseteq V_j \cup \ldots \cup V_{j+m-1}$*

Note that the definition and Lemma 2.12 immediately imply the following:

Remark 2.14. Suppose that every reduced path from a to b contains x and let γ_1, γ_2 be paths from a to x and from x to b respectively. Then the path $\gamma_1\gamma_2$ is reduced if and only if γ_1 and γ_2 are.

If $\gamma = (a, \dots, b)$ is a reduced path and changes direction in $x \in V_k$, then either every reduced path from a to b contains x or the path $(y, x, z) \subseteq \gamma$ can be replaced by an E_{k-1}-path (or by an E_k-path) $(y, \dots z)$ such that $(a, \dots, y, \dots, z, \dots b)$ is still reduced.

Lemma 2.15. *Suppose $\gamma = (a = x_0, \dots, x_s = b)$ is a reduced path from a to b which can be completed into a simple cycle $(a, x_1, \dots, b, \dots, a)$. If γ changes direction in $x \in V_k$, then there is a reduced path $\gamma' = (a = y_0, \dots, y_t = b)$ not containing x such that x is on the E_k-(or E_{k+1}-) path between appropriate elements of γ'.*

Proof. To fix notation let us assume that the neighbours of x in γ are $y, z \in V_{k-1}$. By Lemma 2.12 we may replace the path (y, x, z) by an E_k-path. This path is obviously still reduced and x is on the E_{k+1}-path from y to z. $\square$

Corollary 2.16. *Suppose D is a nice set containing an E_j-path from a to b. Suppose furthermore that $\gamma = (a = y_0, \dots, y_t = b)$ is a reduced path such that the composition with the E_j-path is a simple cycle. If γ changes direction in c, then $c \in D$.*

Using the fact that M_n is ω-saturated and homogeneous for strong substructures we can now describe the algebraic closure:

Lemma 2.17. *The algebraic closure* $\mathrm{acl}(a, b)$ *is the intersection of all nice sets containing a, b.*

Proof. Since there are finite nice sets containing a, b, the intersection of all nice sets containing a, b is certainly finite and invariant over a, b, hence contained in $\mathrm{acl}(a, b)$.

Conversely let D be a nice set containing $\{a, b\}$ and $c \notin D$. If there is no reduced path between a and b changing direction in c, then c has infinitely many conjugates over ab, so $c \notin \mathrm{acl}(ab)$.

So suppose there is a reduced path γ' from a to b changing direction in c. Let $\gamma \subset D$ be a path from a to b. Composing these paths we obtain a simple cycle γ" through c consisting of subpaths $\gamma_1' \subseteq \gamma_1$ of $\gamma' \subseteq \gamma$, respectively,

from some a' to b'. In particular γ_1 is not a flag. We may assume inductively that $\gamma_1' \cap D = \{a', b'\}$. If $a' \in V_j$, then possibly after exchanging the roles of a', b', there is some $x \in \gamma \cap (V_{j-1} \cup V_j)$ closest to b'. Then a', x are connected by an E_j-path entirely contained in D and $(x, \dots b')$ is a flag. Since the E_j-path changes direction in every vertex, we also find an E_{j-1}-path inside D to the next vertex of the flag $(x, \dots b')$. Since γ_1' doesn't meet D except in a', b', we see that $\gamma_1' {}^\frown (b', \dots, x)$ is a reduced path. Now Lemma 2.16 implies $c \in D$, a contradiction. Hence $c \notin \mathrm{acl}(ab)$. ☐

Note that the proof shows in fact the following:

Corollary 2.18. A vertex $x \neq a, b$ is in $\mathrm{acl}(ab)$ if and only if there is a reduced path from a to b that changes direction in x. Hence $\mathrm{acl}(ab) = \{a, b\}$ if and only if a, b is a flag or a and b are not connected. In fact, we have $\mathrm{dcl}(ab) = \mathrm{acl}(ab)$.

Lemma 2.19. *If $g \in \mathrm{acl}(A)$, there exist $a, b \in A$ with $x \in \mathrm{acl}(ab)$.*

Proof. We may assume that A is finite. By induction it suffices to prove that if $d \in \mathrm{acl}(bc), g \in \mathrm{acl}(ad)$, then $g \in \mathrm{acl}(ab) \cup \mathrm{acl}(bc) \cup \mathrm{acl}(ac)$.

By Corollary 2.18 there is a reduced path $\gamma_1 = (b, \dots, d, \dots, c)$ changing direction in d and a reduced path $\gamma_2 = (d, \dots, g, \dots, a)$ changing direction in g. If $\gamma_1 \cup \gamma_2 \in V_i \cup V_{i+1}$ for some i, then clearly either $(a, \dots, g \dots, d, \dots, b)$ or $(a, \dots, g \dots, d, \dots, c)$ is reduced.

Now assume that $\gamma_1 \cup \gamma_2 \in V_i \cup \ldots \cup V_{i+k}$. Clearly we may assume that d is not contained in every reduced path from a to b or in every reduced path from a to c. Furthermore, we may reduce to the case where no vertex of γ_1 is contained in every reduced path from b to c and no vertex of γ_2 is contained in every reduced path from a to d.

By Remark 2.14 we may therefore replace any element $x \in V_{i+k}$ in the interior of γ_1 and γ_2 by an E_{i+k-1} path between its neighbours. If $d \in V_{i+k}$, we may replace d as follows: the neighbour w of d in γ_2 either appears in the E_{i+k-1} path between the neighbours of d in γ_1 or that path extends to w on one of its ends. We may thus replace the paths $(b, \dots, d, \dots, c)$ and $(a, \dots g, \dots, d)$ by reduced paths $(b, b_1, \dots, w, \dots, c_1, c)$ and $(a, a_1, \dots w)$ contained in $V_i \cup \ldots \cup V_{i+k-1}$, except possibly for the endpoints. We may assume that at most $a \in V_{i+k}$ since we may exchange the roles of V_i and V_{i+k} and replace the elements of V_i instead.

By induction assumption, at least one of $(a_1, \dots w, \dots b_1)$ and $(a_1, \dots w, \dots c_1)$ is reduced, and replacing the new E_{i+k-1}-paths by the old E_{i+k}-paths, this

path remains reduced and changes direction in g. If $a, b \in V_{i+k}$, then we may exchange the roles of V_i and V_{i+k} and replace the elements of V_i instead. Thus we see that the path remains reduced when adding the endpoint. $\square$

Proposition 2.20. *For any set A, the algebraic closure* $\mathrm{acl}(A)$ *is the intersection of all nice sets containing it.*

Proof. Clearly, we may restrict ourselves to finite sets A. As before we see that the intersection of all nice sets containing A is contained in $\mathrm{acl}(A)$.

For the converse, assume that c is not in the intersection of all nice sets containing A. If c was in $\mathrm{acl}(A)$, then by Lemma 2.19 there are $a, b \in A$ with $c \in \mathrm{acl}(ab)$. Then by Lemma 2.17 $c \in \bigcap_{a,b \in D} D \subset \bigcap_{A \subseteq D} D$, a contradiction. $\square$

Proposition 2.21. *For any vertex a and set A, there is a flag $C \in \mathrm{acl}(A)$ such that for any $b \in \mathrm{acl}(A)$ there is a reduced path from a to b passing through one of the elements of C.*

The flag C is called the *projection* from a to A and we write $C = \mathrm{proj}(a/A)$. Note that $\mathrm{proj}(a/A) = \emptyset$ if and only if a is not connected to any vertex of $\mathrm{acl}(A)$.

Proof. Let $b_1, b_2 \in \mathrm{acl}(A)$ and let $\gamma_1 = (x_0 = a, \ldots, x_m = b_1)$ and $\gamma_2 = (y_0 = a, \ldots, y_l = b_2)$ be reduced paths such that i, j are minimal possible with $x_i, y_j \in \mathrm{acl}(A)$. By composing the initial segments of γ_1 and γ_2 and reducing we obtain a reduced path from x_i to y_j intersecting $\mathrm{acl}(A)$ only in x_i, y_j since x_i, y_j were chosen at minimal distance from a. By Corollary 2.18 x_i, y_j is a flag. Thus the set of such vertices forms a flag C. $\square$

It is now easy to show the following:

Theorem 2.1. *The theory T_n is ω-stable.*

Proof. Let M be a countable model and let $\bar{d}$ be a tuple from $\overline{M}$. Let $C \in M$ be the finite set of projections from $\bar{d}$ to M. Then the type $\mathrm{tp}(\bar{d}/M)$ is determined by $\mathrm{tp}(\bar{d}/C)$. By Lemmas 2.10 and 2.11, $\bar{d} \cup C$ is contained in a finite strong subset of M_n and for such subsets the quantifier-free type determines the type by Remark 2.10. Hence there are only countably many types over a countable model. $\square$

In fact, it is easy to see directly without counting types that T_n is superstable (see Remark 2.24).

Corollary 2.22. *The free pseudospace has weak elimination of imaginaries.*

Proof. Let a be a vertex and A any set. Then we can choose $\mathrm{Cb}(stp(a/A))$ as the projection of a on A. This is a finite set. $\square$

The following immediate corollary will be very useful:

Corollary 2.23. *The vertex a is independent from A over C if* $\mathrm{proj}(a/AC) \subseteq \mathrm{acl}(C)$*. In particular, a is independent from A over $\emptyset$ if and only if a is not connected to any vertex of* $\mathrm{acl}(A)$*.*

Side remark 2.24. As in [5] we could have defined a notion of independence on models of T_n by saying

$$A \mathop{\smile\!\!\!\!\!\!|}_{C} B$$

if and only if $\mathrm{proj}(a/BC) \subseteq \mathrm{acl}(C)$ for all $a \in \mathrm{acl}(A)$. It is easy to see that this notion of independece satisfies the characterizing properties of forking in stable theories (see [4] Ch. 8) and hence agrees with the usual one. Note that the existence of nonforking extensions follows from the construction of M_n as a Hrushovski limit. Since we have just seen that for any type $\mathrm{tp}(a/A)$ there is a finite set A_0 such that $a \mathop{\smile\!\!\!\!\!\!|}_{A_0} A$ this shows directly (without counting types) that T_n is superstable.

Using this description of forking it is easy to give a list of regular types such that any nonalgebraic type is non-orthogonal to one of these. This is entirely similar to the list given in [1] and we omit the details but will return to this point in Section 5. It is also clear from this description of forking that the geometry on these types is trivial.

3 Ampleness

We now recall the definition of a theory being n-ample:

Definition 3.1. *A theory T eliminating imaginaries is called n-ample if possibly after naming parameters there are tuples $a_0, \ldots a_n$ in M such that the following holds:*

1. *for $i = 0, \ldots n-1$ we have*

$$\mathrm{acl}(a_0, \ldots . a_{i-1}, a_i) \cap \mathrm{acl}(a_0, \ldots . a_{i-1}, a_{i+1}) = \mathrm{acl}(a_0, \ldots . a_{i-1});$$

2. $a_n \not\mathop{\smile}\!\!\!\!\!\!\!\!\;|\;\, a_0$, *and*

3. $a_n \mathop{\smile}\!\!\!\!\!\!\!\!\;|\;_{a_i} a_0 \ldots a_i$ *for* $i = 0, \ldots n-1$.

Theorem 3.1. *The theory T_n is n-ample and any maximal flag $(x_0, \ldots x_n)$ in M_n is a witness for this.*

Proof. This follows immediately from the description of acl in Lemma 2.20 and of forking in Corollary 2.23. □

Theorem 3.2. *The free pseudospace of dimension n is not $n+1$-ample.*

Proof. Suppose towards a contradiction that $a_0, \ldots a_{n+1}$ are witnesses for T_n being $n+1$-ample over some set of parameters A. We have

$$a_{n+1} \underset{A}{\not\perp} a_0,$$

$$a_{n+1} \underset{Aa_i}{\perp} a_0 \ldots a_i, i = 0, \ldots n.$$

By the first condition there are vertices in $\mathrm{acl}(a_0)$ and in $\mathrm{acl}(a_{n+1})$ which are in the same connected component. Put $f_0 = \mathrm{proj}(a_{n+1}/a_0A) \in \mathrm{acl}(a_0A)$ and $f_{n+1} = \mathrm{proj}(f_0/a_{n+1}A) \in \mathrm{acl}(a_{n+1}A)$.

Since

$$a_{n+1} \underset{Aa_i}{\perp} a_0 \ldots a_i, i = 1, \ldots n$$

using Corollary 2.23 we inductively find flags

$$f_i = \mathrm{proj}(f_{n+1}/f_0 f_1 \ldots f_{i-1} a_i A) = \mathrm{proj}(f_{n+1}/a_i A) \in \mathrm{acl}(a_i A), i = 1, \ldots n$$

such that

$$f_{n+1} \underset{f_i}{\perp} f_0 f_1 \ldots f_i.$$

For $i = 1, \ldots n$ we clearly have

$$\mathrm{acl}(f_0, f_1 \ldots . f_{i-1}, f_i) \cap \mathrm{acl}(f_0, \ldots . f_{i-1}, f_{i+1}) \subseteq \mathrm{acl}(a_0, \ldots . a_{i-1} A).$$

By construction, there is a reduced path $\gamma = (f_0, x_1, \ldots x_k = f_{n+1})$ containing a vertex of each of the f_i in ascending order. Since we cannot have a flag containing more than n elements, there must be some vertex x in γ where γ changes direction. For some i we then have $x \in f_{i+1}$ or x occurs in γ

between an element of f_i and an element of f_{i+1}. By Corollary 2.18 we have $x \in \operatorname{acl}(f_i f_{i+1}) \cap \operatorname{acl}(f_i f_{i+2})$. Then

$$x \underset{f_i}{\downarrow} a_0 a_1, \dots a_i A,$$

so $x \notin \operatorname{acl}(a_0 a_1, \dots a_i A)$, a contradiction. $\square$

The proof shows that in fact the following stronger ampleness result holds:

Corollary 3.2. *If $a_0, \dots a_n$ are witnesses for T_n being n-ample, then there are vertices $b_i \in \operatorname{acl}(a_i)$ such that $(b_0, \dots b_n)$ is a flag.*

4 Buildings and the prime model of T_n

We now turn towards constructing the prime model M_n^0 of T_n as a Hrushovski-limit. We will show that M_n^0 is the building associated to a right-angled Coxeter group.

For this purpose we now consider an expansion L'_n of the language L_n by binary function symbols f_k^i. For an L_n-graph A we put $f_k^i(x, y) = z$ if z is the k^{th} element on a unique shortest E_i-path of length at least k from x to y and $z = x$ otherwise.

We say that an L_n-graph A is E_i-connected if the set $V_{i-1}(A) \cup V_i(A)$ is connected.

Definition 4.1. *Let $\mathcal{K}'$ be the class of finite L'_n-graphs $A \in \mathcal{K}$ which are E_i-connected for $i = 1, \dots n$ and additionally satisfy the following condition:*

6. *If $a \in A$ is of type V_j, then the residue $R(a)$ is E_i-connected for $i = 1, \dots n$.*

Note that $\mathcal{K}'$ is closed under finitely generated substructures by the choice of language.

Definition 4.2. *Let A be a finite L'_n-graph which is E_i-connected for $i = 1, \dots n$. The following extensions are called elementary strong extensions of A:*

1. *add a vertex of type P or E to A which is connected to at most one vertex of A and such that the extension is still E_i-connected for all $i = 1, \dots n$.*

2. *If (x, y, z) is a dense flag in A, add a vertex y' of the same type as z to A such that (x, y', z) is a flag.*

3. *if $|A| \leq 1$ contains no line, add a vertex of appropriate type which is connected to the vertex of A if $A \neq \emptyset$.*

Again we write $A \leq B$ if B arises from A by finitely many elementary strong extensions.

We next show that $(\mathcal{K}'_n, \leq)$ has the amalgamation property for $\leq$-extensions.

Lemma 4.3. *If A contains a flag of type $(V_1, \ldots V_{n-1})$ and $A \leq B, C$ are in $\mathcal{K}'_n$, then $D := B \otimes_A C \in \mathcal{K}'_n$ and $B, C \leq D$.*

Proof. Clearly, $B, C \leq D$ and D is E_i-connected for all $i = 1, \ldots n$ since A contains a flag. To see that $D \in \mathcal{K}$, note that if $B \in \mathcal{K}$ and B' is an elementary strong extension of B, then also $B' \in \mathcal{K}$. $\square$

This shows that the class $(\mathcal{K}, \leq)$ has a Hrushovski limit M_n^0. Clearly, M_n^0 is E_i-connected for $i = 1, \ldots n$ and since any two vertices of M_n^0 lie in a maximal flag, it follows that M_n^0 is in fact n-connected. Note that an L'_n-substructure of M_n^0 is automatically nice, see Remark 2.10.

The same proof as in the case of M_n shows the first part of the following proposition:

Proposition 4.4. *The Hrushovski limit M_n^0 is a model of T_n and M_n^0 is the unique countable model of T_n which is E_i-connected for $i = 1, \ldots n$ and such that every vertex is contained in a maximal flag.*

(Note that in [1] the corresponding Remark 3.6 of uses Lemma 3.2, which is not correct as phrased there: M_n^0 and $M_n^0 \cup \{a\}$ with a an isolated point are not isomorphic, but satisfy the assumptions of Remark 3.6.)

The uniqueness part of Proposition 4.4 follows directly from the following theorem and Proposition 5.1 of [3] which states that this type of building is uniquely determined by its associated Coxeter group and the cardinality of the residues.

Theorem 4.1. *M_n^0 is a building of type $A_{\infty,n+1}$ all of whose residues have cardinality $\aleph_0$.*

Recall the following definitions (see e.g. [2]).
Let W be the Coxeter group

$$W = \langle t_0, \dots t_n \colon t_i^2 = (t_i t_k)^2 = 1, i, k = 0 \dots n, |k - i| \geq 2 \rangle,$$

whose associated diagram we call $A_{\infty,n+1}$.

Definition 4.5. *A building of type $A_{\infty,n+1}$ is a set Δ with a* Weyl distance function $\delta : \Delta^2 \to W$ *such that the following two axioms hold:*

1. *For each $s \in S := \{t_i, i = 0, \dots n\}$, the relation $x \sim_s y$ defined by $\delta(x, y) \in \{1, s\}$ is an equivalence relation on Δ and each equivalence class of $\sim_s$ has at least 2 elements.*

2. *Let $w = r_1 r_2 \dots r_k$ be a shortest representation of $w \in W$ with $r_i \in S$ and let $x, y \in \Delta$. Then $\delta(x, y) = w$ if and only if there exists a sequence of elements $x, x_0, x_1, \dots, x_k = y$ in Δ with $x_{i-1} \neq x_i$ and $\delta(x_{i-1}, x_i) = r_i$ for $i = 1, \dots, k$.*

A sequence as in 2. is called a *gallery of type* $(r_1, r_2, \dots, r_k)$. The gallery is called reduced if the word $w = r_1 r_2, \dots, r_k$ is reduced, i.e. a shortest representation of w.

We now show how to consider M_n^0 as a building of type $A_{\infty,n+1}$.

Proof. (of Theorem 4.1) We extend the set of edges of the $n+1$-coloured graph M_n^0 by putting edges between any two vertices that are incident in the sense of Definition 2.11. In this way, flags of M_n^0 correspond to a complete subgraph of this extended graph, which thus forms a simplicial complex. A maximal simplex consists of $n+1$ vertices each of a different type V_i. (Such a simplex is called a *chamber*.) Let Δ be the set of maximal simplices in this graph. Define $\delta : \Delta^2 \to W$ as follows:
Put $\delta(x, y) = t_i$ if and only if the flags x and y differ exactly in the vertex of type V_i. Extend this by putting $\delta(x, y) = w$ for a reduced word $w = r_1 r_2 \dots r_k$ if and only if there exists a sequence of elements $x = x_0, x_1, \dots, x_k = y$ in Δ with $x_{i-1} \neq x_i$ and $\delta(x_{i-1}, x_i) = r_i$ for $i = 1, \dots, k$.
Clearly, with this definition of δ, the set Δ satisfies the first condition of Definition 4.5. In fact, for all $s \in S$ every equivalence class $\sim_s$ has cardinality $\aleph_0$.
We still need to show that δ is well-defined, i.e. if we have to show the following for any $x, y \in \Delta$: if there are reduced galleries $x_0 = x, x_1, \dots, x_k = y$

and $y_0 = x, y_1, \ldots, y_m = y$ of type $(r_1, r_2, \ldots, r_k)$ and $(s_1, \ldots s_m)$, respectively, then in W we have $r_1 r_2 \ldots r_k = s_1 \ldots s_m$. Equivalentely, we will show the following, which completes the proof of Theorem 4.1:

Claim: There is no reduced gallery $a_0, a_1, \ldots, a_k = a_0$ for $k > 0$ in M_n^0.

Proof of Claim. Suppose otherwise. Let $a_0, a_1, \ldots, a_k = a_0$ be a reduced gallery of type $(r_1, \ldots r_k)$ for some $k > 0$. Note that the flags a_i and a_{i+1} contain the same vertex of type V_j as long as $r_i \neq t_j$.

Now consider the sequence of vertices of type V_n and V_{n-1} occurring in this gallery. Since $V_n \cup V_{n-1}$ contains no cycles, the sequence of vertices of type V_n and V_{n-1} occurring in this gallery will be of the form

$$(x_1, y_1, x_2, y_2, \ldots, x_i, y_i, x_i, y_{i-1}, \ldots y_1, x_1) \tag{1}$$

with $x_i \in V_n, y_i \in V_{n-1}$ and x_i a neighbour of y_i and y_{i-1} in the original graph. This implies that at some place in the gallery type there are two occurrences of t_n which are not separated by an occurrence of t_{n-1} (or conversely). Since t_n commutes with all t_i for $i \neq n-1$ and the word $r_1 \ldots r_k$ is reduced, there are two occurrences of t_{n-1} which are not separated by an occurrence of t_n, say $r_j, r_{j+m} = t_{n-1}$ with $r_{j+1}, \ldots, r_{j+m-1} \neq t_n$.

We now consider the gallery $a_j, \ldots a_{j+m}$ of type $(r_j = t_{n-1}, r_{j+1}, \ldots, r_{j+m} = t_{n-1})$. Notice that by (1), the flags a_j and a_{j+m} have the same V_n and the same V_{n-1} vertex. Since M_n^0 does not contain any E_{n-1}-cycles, the sequence of V_{n-1}- and V_{n-2}-vertices appearing in this sequence must again be of the same form as in (1). Exactly as before we find two occurrences[1] of t_{n-2} in the gallery type of $a_j, \ldots a_{j+m}$ which are not separated by an occurrence of t_{n-1}. Continuiung in this way, we eventually find two occurrences of t_1 which are not separated by any t_i. Since $t_1^2 = 1$ this contradicts the assumption that the gallery be reduced.

□

The proof shows in fact the following:

Corollary 4.6. *A model of T_n is a building of type $A_{\infty,n+1}$ if and only if it is E_i-connected for all i and every vertex is contained in a maximal flag.*

Theorem 4.2. *The building M_n^0 is the prime model of T_n*

[1] If t_{n-2} does not occur in the type of the gallery, this would contradict the assumption that the type is reduced since t_{n-1} commutes with all t_i for $i \neq n, n-2$.

Proof. To see that M_n^0 is the prime model of T_n note that for any flags $C_1, C_2 \in M_n^0$ and gallery $C_1, = x_0, \ldots, x_k = C_2$ the set of vertices occuring in this gallery is E_i-connected for all i. Hence by Remark 2.10 its type is determined by the quantifier-free type.

Thus, given a maximal flag M in any model of T_n and a maximal flag c_0 of M_n^0 we can embed M_n^0 into M by moving along the galleries of M_n^0. $\square$

5 Ranks and types

Recall that for vertices $x, y \in M_n^0$ with $x \in V_i, y \in V_j$ the *Weyl-distance* $\delta(x, y)$ equals $w \in W$ if there are flags C_1, C_2 containing x, y, respectively, with $\delta(C_1, C_2) = w'$ and such that w is the shortest representative of the double coset $\langle t_k \colon k \neq i\rangle^W w' \langle t_k \colon k \neq j\rangle^W$ (where as usual $\langle X\rangle^W$ denotes the subgroup of W generated by X).

The following is clear:

Proposition 5.1. *The theory T_n has quantifier elimination in a language containing predicates $\delta_w^{i,j}$ for Weyl distances between vertices of type V_i and of type V_j.*

For any small set A in a large saturated model we have the following kinds of regular types:

(I) $\mathrm{tp}(a/A)$ where $a \in V_i$ is not connected to any element in $\mathrm{acl}(A)$

(II) $\mathrm{tp}(a/A)$ where $a \in V_i$ is incident with some $b \in \mathrm{acl}(A) \cap V_j$ but not connected in $R(b)$ to any vertex in $\mathrm{acl}(A) \cap R(b)$.

(III) $\mathrm{tp}(a/A)$ where $a \in V_i$ is incident with some $x, y \in \mathrm{acl}(A)$ such that (x, a, y) is a flag with $x \in V_k, y \in V_j$; and as a special case of this we have

(IV) $\mathrm{tp}(a/A)$ where $a \in V_i$ has neighbours $x, y \in \mathrm{acl}(A)$ such that (x, a, y) is a (necessarily dense) flag.

By quantifier elimination any of these descriptions determines a complete type. Using the description of forking in Corollary 2.23 one sees easily that each of these types is regular and trivial.

Clearly, any type in (IV) has U-rank 1 and in fact Morley rank 1 by quantifier elimination. It also follows easily that $\mathrm{MR}(a/A) = \omega^n$ if $\mathrm{tp}(a/A)$ is

as in (I). In case (II) we find that $\mathrm{MR}(a/A) = \omega^{n-j-1}$ or $\mathrm{MR}(a/A) = \omega^{j-1}$ depending on whether or not $i < j$. In case (II) we have $\mathrm{MR}(a/A) = \omega^{|k-j|-2}$.

Just as in [1] we obtain:

Lemma 5.2. *Any regular type in T_n is non-orthogonal to a type as in (I) or as in (IV).*

Proof. Let $p = \mathrm{tp}(b/\operatorname{acl}(B))$. If b is not connected to $\operatorname{acl}(B)$, then p is as in (I), so we may assume that $\mathrm{proj}(b/B) = C \neq \emptyset$. Let a be a vertex on a short path from b to C incident with an element of C. Then by Corollary 2.23 we see that p is non-orthogonal to $\mathrm{tp}(a/C)$. $\square$

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The free pseudospace is n-ample, but not $(n+1)$-ample

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Abstract

We give a uniform construction of free pseudospaces of dimension n extending work in [1]. This yields examples of ω-stable theories which are n-ample, but not $n+1$-ample. The prime models of these theories are buildings associated to certain right-angled Coxeter groups.

1 Introduction

In the investigation of geometries on strongly minimal sets the notion of ampleness plays an important role. Algebraically closed fields are n-ample for all n and it is not known whether there are strongly minimal sets which are n-ample for all n and do not interpret an infinite field. Obviously, one way of proving that no infinite field is interpretable in a theory is by showing that the theory is *not* n-ample for some n.

In [1], Baudisch and Pillay constructed a free pseudospace of dimension 2. Its theory is ω-stable (of infinite rank) and 2-ample. F. Wagner posed the question whether this example was 3-ample or not.

In Section 2 we give a uniform construction of a free pseudospace of dimension n and show that it is n-ample, but not $n+1$-ample. It turns out that the theory of the free pseudospace of dimension n is the first order theory of a Tits-building associated to a certain Coxeter diagram and we will investigate this connection in Section 4.

In the final section we determine the orthogonality classes of regular types.

The construction given here is quite similar to the one given by Evans in [2] for a stable theory which is n-ample for all n, but does not interpret an

infinite group. In contrast to the examples constructed by Evans, our theory is trivial and no infinite group is definable. Baudisch, Pizarro and Ziegler informed me that they also showed that the examples in [1] are not 3-ample. In thank Anand Pillay for pointing out a mistake in an earlier version of this paper and Linus Kramer, in particular for providing reference [4].

2 Construction and results

Fix a natural number $n \geq 1$. Let L_n be the language for $n+1$-coloured graphs containing predicates $V_i, i = 0, \dots n$ and an edge relation E. If $x \in V_i$ we also say that x is of level i.

By an L_n-graph we mean an $n+1$-coloured graph with vertices of types $V_i, i = 0, \dots n$ and an edge relation $E \subseteq \bigcup_{i=1,\dots n} V_{i-1} \times V_i$. We say that a path in this graph is of type E_i if all its vertices are in $V_{i-1} \cup V_i$ and of type $E_i \cup \dots \cup E_{i+j}$ if all its vertices are in $V_{i-1} \cup \dots \cup V_{i+j}$

The free pseudospaces will be modeled along the lines of a projective space, i.e. we will think of vertices of type V_i as i-dimensional spaces in a free pseudospace. Therefore we extend the notion of incidence as follows:

Definition 2.1. *1. We say that a vertex x_i of type V_i is* incident *to a vertex x_j of type V_j if there are vertice x_l of type $V_l, l = i+1 \dots j$ such that $E(x_{l-1}, x_l)$ holds. In this case the sequence $(x_i, \dots x_j)$ is called a* dense flag. *A* flag *is a sequence of vertices $(x_1, \dots x_k)$ in which any two vertices are incident.*

2. *The* residue *$R(x)$ of a vertex x is the set of vertices incident with x.*

3. *We say that two vertices x and y* intersect in the vertex *z and write $z = x \wedge y$ if the set of vertices of type V_0 incident with x and y is exactly the set of vertices of type V_0 incident with z. If there is no vertex of type V_0 incident to x and y, we say that x and y intersect in the empty set.*

4. *We say that two vertices x and y* generate the vertex *z and write $z = x \vee y$, if the set of vertices of type V_n incident with x and y is exactly the set of vertices of type V_n incident with z. If there is no vertex of type V_n incident to x and y, we say that x and y generate the empty set.*

5. *A* simple cycle *is a cycle without repetitions.*

We now give an inductive definition of a free pseudospace of dimension n:

Definition 2.2. *A free pseudospace of dimension* 1 *is a free pseudoplane, i.e. an L_1-graph which does not contain any cycles and such that any vertex has infinitely many neighbours.*

Assume that a free pseudospace of dimension $n-1$ has been defined. Then a free pseudospace of dimension n is an L_n-graph such that the following holds:

$(\Sigma 1)_n$ *(a) The set of vertices of type $V_0 \cup \ldots \cup V_{n-1}$ is a free pseudospace of dimension $n-1$.*

(b) The set of vertices of type $V_1 \cup \ldots \cup V_n$ is a free pseudospace of dimension $(n-1)$.

$(\Sigma 2)_n$ *(a) For any vertex x of type V_0, $R(x)$ is a free pseudospace of dimension $(n-1)$.*

(b) For any vertex x of type V_n, $R(x)$ is a free pseudospace of dimension $(n-1)$.

$(\Sigma 3)_n$ *(a) Any two vertices x and y intersect in some vertex z or the emptyset.*

(b) Any two vertices x and y generate some vertex z or the emptyset.

$(\Sigma 4)_n$ *(a) If a is a vertex of type V_n and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k not contained in $R(a)$, then there is an E_{n-1}-path from b to b' in $R(a)$ of length at most $k-1$.*

(b) If a is a vertex of type V_0 and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k not contained in $R(a)$, then there is an E_2-path from b to b' in $R(a)$ of length at most $k-1$.

Let T_n denote the L_n-theory expressing these axioms.

Note that the inductive nature of the definition immediately has the following consequences:

1. The induced subgraph on $V_j \cup \ldots \cup V_{j+m}$ is a free pseudospace of dimension m.

2. If a is a vertex of type V_{j+m} and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k contained in $V_j \cup \ldots \cup V_{j+m}$, then there is an $E_{j+1} \cup \ldots \cup E_{j+m-2}$-path from b to b' of length at most $k-1$ in $R(a)$ all of whose V_j-vertices appear in γ.

3. The notion of a free pseudospace of dimension n is *self-dual*: if we put $W_i = V_{n-i}, i = 0, \ldots n$, then $W_0, \ldots W_n$ with the same set of edges is again a free pseudospace of dimension n.

Our first goal is to show that T_n is consistent and complete.

Definition 2.3. *Let $\mathcal{K}_n$ be the class of finite L_n-graphs A such that the following holds*

1. *A does not contain any E_i-cycles for $i = 1, \ldots n$.*

2. *If $a \neq a'$ are in A, they intersect in a vertex of A or the emptyset.*

3. *If $a \neq a'$ are in A, they generate a vertex of A or the emptyset.*

4. *If (b, a, b') is a path with $a \in V_i, b, b' \in V_{i-1}$, and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k not contained in $R(a)$, then there is some E_{i-1}-path from b to b' of length at most $k-1$ in $R(a)$.*

5. *If (b, a, b') is a path with $a \in V_i, b, b' \in V_{i+1}$, and $\gamma = (a, b, \ldots, b', a)$ is a simple cycle of length k not contained in $R(a)$, then there is some E_{i+2}-path from b to b' of length at most $k-1$ in $R(a)$.*

Definition 2.4. *Let A be a finite L_n-graph. The following extensions are called* 1*-point strong extensions of A:*

1. *add a vertex of any type to A which is connected to at most one vertex of A of an appropriate type.*

2. *If (x, y, z) is a dense flag in A, add a vertex of the same type as y to A which is connected to both x and z.*

We write $A \leq B$ if B arises from A by finitely many 1*-point strong extensions.*

We next show that $(\mathcal{K}, \leq)$ has the amalgamation property for strong extensions. This will be enough to obtain a strong limit which is well-defined up to automorphism (see [8]).

For any finite L_n-graphs $A \subseteq B, C$ we denote by $B \otimes_A C$ the *free amalgam* of B and C over A, i.e. the graph on $B \cup C$ containing no edges between elements of $B \setminus A$ and $C \setminus A$.

Lemma 2.5. *If $A \leq B, C$ are in $\mathcal{K}_n$, then $D := B \otimes_A C \in \mathcal{K}$ and $B, C \leq D$.*

Proof. Clearly, $B, C \leq D$. To see that $D \in \mathcal{K}_n$, note that if $B \in \mathcal{K}_n$ and B' is an 1-point strong extension of B, then also $B' \in \mathcal{K}_n$. This is clear for strong extensions of type 1. For strong extensions of type 2. suppose that (b, a, b') is a path with $a \in V_i, b, b' \in V_{i-1}$, and $\gamma = (a, b, \ldots, b', a) \subset B'$ is an $E_i \cup \ldots \cup E_{i-j}$-path of length k containing the new vertex y. Since the new vertex has exactly two neighbours y_1, y_2, this implies that the vertex is of type V_m for some $i - j \leq m \leq i$ and (y_1, y, y_2) is contained in γ. By construction of strong extensions, there is some $z \in B$ such that (y_1, z, y_2) is a path. Hence we may replace all occurrences of y in γ by z. Then γ is contained in B and we find the required path in $R(a)$ with all V_{i-j}-vertices occurring in γ. □

This shows that the class $(\mathcal{K}_n, \leq)$ has a strong Fraïssé limit M_n. Here we say as usual that a subset A of M_n is strong in M_n if $A \cap B \leq B$ for any finite set $B \subset M_n$.

Proposition 2.6. *The Hrushovski limit M_n is a model of T_n.*

Proof. By construction, $V_i \cup \ldots \cup V_{i+j}$ satisfies $(\Sigma 3)_j$ and $(\Sigma 4)_j$ for any i, j. In particular, M_n satisfies $(\Sigma 3)_n$ and $(\Sigma 4)_n$.

$(\Sigma 1)_n$: In order to show that M satisfies $(\Sigma 1)_n$, we first note that $V_i \cup V_{i+1}$ is a free pseudoplane for all $i = 0, \ldots n-1$. Assume inductively that $V_j \cup \ldots \cup V_{j+i}$ is a free pseudospace of dimension i. To see that $V_j \cup \ldots \cup V_{j+i+1}$ is a free pseudospace of dimension $i+1$, we need only verify $(\Sigma 2)_{i+1}$. Hence we have to show that for $a \in V_j$ the residue $R(a) \cap (V_j \cup \ldots \cup V_{j+i+1})$ is a free pseudospace of dimension i. We know by induction that $R(a) \cap (V_j \cup \ldots \cup V_{j+i})$ is a free pseudospace.

Clearly,

$$R(a) \cap (V_{j+1} \cup \ldots \cup V_{j+i+1}) = \bigcup \{R(b) \cap (V_{j+1} \cup \ldots \cup V_{j+i+1}) \colon b \in V_{j+1}, E(a, b)\}.$$

For each neighbour $b \in V_{j+1}$ of a, the set $R(b) \cap (V_{j+1} \cup \ldots \cup V_{j+i+1})$ is a free pseudospaces of dimension $i-1$ by induction. Since $(V_{j+1} \cup \ldots \cup V_{j+i+1})$ is a free pseudospace of dimension i, $(\Sigma 2)_{i+1}$ follows from the induction hypothesis. Hence $V_0 \cup \ldots \cup V_{n-1}$ and $V_1 \cup \ldots \cup V_n$ are free pseudospaces of dimension $n-1$.

$(\Sigma 2)_n$: The proof of $(\Sigma 2)_n$ is similar. □

We say that a model M of T_n is $\mathcal{K}_n$-saturated if for all finite $A \leq M$ and strong extensions C of A with $C \in \mathcal{K}_n$ there is a strong embedding of C into M fixing A elementwise. Clearly, by construction, M_n is $\mathcal{K}_n$-saturated.

Lemma 2.7. *An L_n-structure M is an ω-saturated model of T_n if and only if M is $\mathcal{K}_n$-saturated.*

Proof. Let M be an ω-saturated model of T_n. To show that M is $\mathcal{K}_n$-saturated, let $A \leq M$ and $A \leq B \in \mathcal{K}_n$. By induction we may assume that B is an 1-point strong extension of A and by ω-saturation it is easy to see that B can be imbedded over A into M. Conversely assume that M is $\mathcal{K}_n$-saturated. Since M is partially isomorphic to M_n, it is a model of T_n. Choose an ω-saturated $M' \equiv M$. Then by the above M' is $\mathcal{K}_n$-saturated. So M' and M are partially isomorphic, which implies that M is ω-saturated. □

Corollary 2.8. *The theory T_n is complete.*

Proof. Let M be a model of T_n. In order to show that M is elementarily equivalent to M_n choose an ω-saturated $M' \equiv M$. By Lemma 2.7, M' is $\mathcal{K}_n$-saturated. Now M' and M_n are partially isomorphic and therefore elementarily equivalent. □

We will see in Section 4 that T_n is the theory of the building of type $A_{\infty,n+1}$ with infinite valencies.

Definition 2.9. *Following* [1] *we call a subset A of a model M of T_n* nice *if*

1. *any E_i-path between elements of A lies entirely in A and*

2. *if $a, b \in A$ are connected by a path in M there is a path from a to b inside A.*

Remark 2.10. Note that a subset A of M_n is strong in M_n if and only if it is nice. (This follows immediately from the definition of strong extension.)

We now work in a very saturated model $\overline{M}$ of T_n.

Lemma 2.11. *If A is a finite set, there is a nice finite set B containing A.*

Proof. Since single vertices are nice it suffices to prove the following

Claim: *If A is nice and a arbitrary, then there is a nice finite set B containing $A \cup \{a\}$.*

Proof of Claim: Of course we may assume $a \notin A$. If there is no path from a to A, clearly $A \cup \{a\}$ is nice. Hence we may also assume that there is some path $\gamma = (a = x_0, \dots b)$ for some $b \in A$ and $\gamma \cap A = \{b\}$. It therefore suffices to prove the claim for the case where a has a neighbour in A. If a has two neighbours $x, y \in A$ then (x, a, y) is a dense flag and $A \cup \{a\}$ is nice.

Now assume that $a \in V_i$ has a unique neighbour of type V_{i+1} in A. (The other case then follows by self-duality.) If the E_i-connected component of a does not intersect A, then again $A \cup \{a\}$ is nice. Otherwise there is some E_i-path $\gamma = (x_0 = a, \dots x_m = b)$ in M_n with $\gamma \cap A = \{b\}$. If for some V_{i-1}-vertex x_k of γ there is an E_{i-1}-path to some $c \in A$, then the E_i-path from c to b extends $(x_k, \dots x_m = b)$ and is entirely contained in A since A is nice. Since $\gamma \cap A = \{b\}$, no such x_k exists implying that $A \cup \gamma$ is nice. $\square$

Corollary 2.12. *The algebraic closure* $\operatorname{acl}(A)$ *contains the intersection of all nice sets containing A.*

Let us say that γ *changes direction in x_i* if $x_i \in V_j$ and either $x_{i-1}, x_{i+1} \in V_{j-1}$ or $x_{i-1}, x_{i+1} \in V_{j-1}$ for some j. Clearly a path which doesn't change direction is a dense flag.

Definition 2.13. *We call a path $\gamma = (x_0, \dots x_k) \subseteq V_j \cup \dots \cup V_{j+m}$* reduced *if the following holds:*

1. *if $m = 1$ the path γ is reduced if it does not contain any repetition.*
2. *any path $(x_{i-1}, x_i, z_1, \dots, z_t, x_{i+k}, x_{i+k+1})$ contained in $V_j \cup \dots \cup V_{j+m-1}$ or in $V_{j+1} \cup \dots \cup V_{j+m}$ is reduced if $(x_i, z_1, \dots, z_t, x_{i+k})$ is reduced.*

Note that the definition immediately implies the following:

Remark 2.14. Suppose that every reduced path from a to b contains x and let γ_1, γ_2 be paths from a to x and from x to b respectively. Then the path $\gamma_1\gamma_2$ is reduced if and only if γ_1 and γ_2 are.

Using the fact that M_n is ω-saturated we can now describe the algebraic closure:

Lemma 2.15. *A vertex $x \neq a, b$ is in $\mathrm{acl}(ab)$ if and only if there is a reduced path from a to b that changes direction in x. Hence $\mathrm{acl}(ab) = \{a, b\}$ if and only if a, b is a flag or a and b are not connected.*

We have in fact $\mathrm{acl}(ab) = \mathrm{dcl}(ab)$.

Proof. If there is no reduced path between a and b changing direction in c, then c has infinitely many conjugates over ab, hence $c \notin \mathrm{acl}(ab)$. So suppose there is a reduced path from a to b changing direction in c. Let $C_{a,b}$ the set of all vertices y such that there is a reduced path $(a, \ldots, x, y, z, \ldots b)$ changing direction in y and such that for $x, z \in V_j$ are not connected by both an E_j and an E_{j+1}-path.

We claim that $C_{a,b}$ is a finite set. Let $\gamma = (a, \ldots b)$ be a reduced path. For any $y \in C_{a,b} \setminus \gamma$ let $\gamma_y = (a, \ldots, x, y, x, \ldots b)$ be a reduced path witnessing that $y \in C_{a,b}$. Composing suitable pieces of γ and γ_y we obtain a simple cycle γ_y" changing direction in y. Since $y \in C_{a,b}$ we have $\gamma_y'' \subset R(y)$ by ($\Sigma 4$). For $y_1 \neq y_2 \in V_j \cap C_{a,b}$ the paths γ_{y_1}" $\cap\, \gamma$ and γ_{y_2}" $\cap\, \gamma$ must be disjoined since otherwise we obtain a simple cycle changing direction in y_1 and y_2, but not contained in $R(y_1)$ which would contradict $y_1 \in C_{a,b}$.

Clearly $C_{a,b}$ is invariant under all automorphisms fixing a, b and so $C_{a,b} \subseteq \mathrm{acl}(ab)$.

Now consider a reduced path γ from a to b changing direction in c. We may inductively assume that $\gamma \cap \mathrm{acl}(ab) = \{a, b\}$ since otherwise we may replace a, b by some $a', b' \in \mathrm{acl}(ab) \cap \gamma$ and consider the piece of γ containing c. In particular $\gamma \cap\ C_{a,b} = \emptyset$.

It is therefore sufficient to prove that any reduced path $\gamma = (a, \ldots, b)$ with $\gamma \cap \mathrm{acl}(ab) = \{a, b\}$ is a flag. We do induction on the number of levels involved. Clearly, if $\gamma \subseteq V_j \cup V_{j+1}$ for some j, then $\gamma \subseteq \mathrm{acl}(ab)$ since such a path is unique. Hence $\gamma = (a, b)$. Now suppose that $\gamma = (a, a_1, \ldots b) \subseteq V_j \cup \ldots \cup V_{j+m}$. We claim that $a \in V_j \cup V_{j+m}$: otherwise we may replace all paths (x, c, y) where γ changes direction by E_k paths for appropriate k and reduce the number of levels of γ (since $\gamma \cap C_{a,b} = \emptyset$). By induction this new path is a flag, which clearly is impossible. Hence $a \in V_j \cup V_{j+m}$ and we may apply the same consideration to the subpath of γ starting at a_1. By induction $(a_1, \ldots b)$ and hence γ are flags. $\square$

In Section 4 we will see that in the prime model the algebraic closure will be described by reduced words in the Coxeter group associated to the building.

Proposition 2.16. *If $g \in \mathrm{acl}(A)$, there exist $a, b \in A$ with $x \in \mathrm{acl}(ab)$.*

Proof. We may assume that A is finite. By induction it suffices to prove that if $d \in \mathrm{acl}(bc)$, $g \in \mathrm{acl}(ad)$, then $g \in \mathrm{acl}(ab) \cup \mathrm{acl}(bc) \cup \mathrm{acl}(ac)$.

By Lemma 2.15 there is a reduced path $\gamma_1 = (b, \ldots, d, \ldots, c)$ changing direction in d and a reduced path $\gamma_2 = (d, \ldots, g, \ldots, a)$ changing direction in g. If $\gamma_1 \cup \gamma_2 \in V_i \cup V_{i+1}$ for some i, then clearly either $(a, \ldots, g \ldots, d, \ldots, b)$ or $(a, \ldots, g \ldots, d, \ldots, c)$ is reduced.

Now assume that $\gamma_1 \cup \gamma_2 \in V_i \cup \ldots \cup V_{i+k}$. Clearly we may assume that d is not contained in every reduced path from a to b or in every reduced path from a to c. Furthermore, we may reduce to the case where no vertex of γ_1 is contained in every reduced path from b to c and no vertex of γ_2 is contained in every reduced path from a to d.

By symmetry we may assume that $d \notin V_{i+k}$. We may then replace any element $x \in V_{i+k}$ in the interior of γ_1 and γ_2 by a reduced path in $R(x)$ between its neighbours. If $a, b, c \in V_{i+k}$, then we may extend γ_1, γ_2 by vertices in V_{i+k-1} to reduced paths containing a, b, c in its interior and also replace a, b, c. We may thus replace the paths $(b, \ldots, d, \ldots, c)$ and $(a, \ldots g, \ldots, d)$ by reduced paths $(b_1, \ldots, d, \ldots, c_1)$ and $(a_1, \ldots d)$ contained in $V_i \cup \ldots \cup V_{i+k-1}$.

By induction assumption, at least one of $(a_1, \ldots d, \ldots b_1)$ and $(a_1, \ldots d, \ldots c_1)$ is reduced, and replacing the new pieces of the path by the old ones, this path remains reduced and changes direction in g. $\square$

Proposition 2.17. *For any vertex a and set A, there is a flag $C \in \mathrm{acl}(A)$ such that for any $b \in \mathrm{acl}(A)$ there is a reduced path from a to b passing through one of the elements of C.*

The flag C is called the *projection* from a to A and we write $C = \mathrm{proj}(a/A)$. Note that $\mathrm{proj}(a/A) = \emptyset$ if and only if a is not connected to any vertex of $\mathrm{acl}(A)$.

Proof. Let $b_1, b_2 \in \mathrm{acl}(A)$ and let $\gamma_1 = (x_0 = a, \ldots, x_m = b_1)$ and $\gamma_2 = (y_0 = a, \ldots, y_l = b_2)$ be reduced paths such that i, j are minimal possible with $x_i, y_j \in \mathrm{acl}(A)$. By composing the initial segments of γ_1 and γ_2 and reducing we obtain a reduced path from x_i to y_j intersecting $\mathrm{acl}(A)$ only in x_i, y_j since x_i, y_j were chosen at minimal distance from a. By Lemma 2.15 x_i, y_j is a flag. Thus the set of such vertices forms a flag C. $\square$

It is now easy to show the following:

Theorem 2.1. *The theory T_n is ω-stable.*

Proof. Let M be a countable model and let $\bar{d}$ be a tuple from $\overline{M}$. Let $C \in M$ be the finite set of projections from $\bar{d}$ to M. Then the type $\mathrm{tp}(\bar{d}/M)$ is determined by $\mathrm{tp}(\bar{d}/C)$. By Lemmas 2.10 and 2.11, $\bar{d} \cup C$ is contained in a finite strong subset of M_n and for such subsets the quantifier-free type determines the type by Remark 2.10. Hence there are only countably many types over a countable model. □

In fact, it is easy to see directly without counting types that T_n is superstable (see Remark 2.20).

Corollary 2.18. *The free pseudospace has weak elimination of imaginaries.*

Proof. Let a be a vertex and A any set. Then we can choose $\mathrm{Cb}(stp(a/A))$ as the projection of a on A. This is a finite set. □

The following immediate corollary will be very useful:

Corollary 2.19. *The vertex a is independent from A over C if* $\mathrm{proj}(a/AC) \subseteq \mathrm{acl}(C)$. *In particular, a is independent from A over $\emptyset$ if and only if a is not connected to any vertex of* $\mathrm{acl}(A)$.

Side remark 2.20. As in [7] we could have defined a notion of independence on models of T_n by saying

$$A \underset{C}{\downarrow} B$$

if and only if $\mathrm{proj}(a/BC) \subseteq \mathrm{acl}(C)$ for all $a \in \mathrm{acl}(A)$. It is easy to see that this notion of independece satisfies the characterizing properties of forking in stable theories (see [6] Ch. 8) and hence agrees with the usual one. Note that the existence of nonforking extensions follows from the construction of M_n as a Hrushovski limit. Since we have just seen that for any type $\mathrm{tp}(a/A)$ there is a finite set A_0 such that $a \downarrow_{A_0} A$ this shows directly (without counting types) that T_n is superstable.

Using this description of forking it is easy to give a list of regular types such that any nonalgebraic type is non-orthogonal to one of these. This is entirely similar to the list given in [1] and we omit the details but will return to this point in Section 5. It is also clear from this description of forking that the geometry on these types is trivial.

3 Ampleness

We now recall the definition of a theory being n-ample given by Pillay in [5].

Definition 3.1. *A theory T eliminating imaginaries is called n-ample if possibly after naming parameters there are tuples $a_0, \dots a_n$ in M such that the following holds:*

1. *for $i = 0, \dots n-1$ we have*

$$\operatorname{acl}(a_0, \dots . a_{i-1}, a_i) \cap \operatorname{acl}(a_0, \dots . a_{i-1}, a_{i+1}) = \operatorname{acl}(a_0, \dots . a_{i-1});$$

2. *$a_n \not\perp a_0$, and*

3. *$a_n \perp_{a_i} a_0 \dots a_i$ for $i = 0, \dots n-1$.*

Remark 3.2. *In [2], Evans requires the slightly more natural condition*

3'. *$a_n a_{n-1} \dots a_{i+1} \perp_{a_i} a_0 \dots a_{i-1}$ for $i = 0, \dots n-1$.*

Theorem 3.1. *The theory T_n is n-ample (in the sense of Evans' defintion) and any maximal flag $(x_0, \dots x_n)$ in M_n is a witness for this.*

Proof. This follows immediately from the description of acl in Lemma 2.15 and of forking in Corollary 2.19. □

Theorem 3.2. *The free pseudospace of dimension n is not $n+1$-ample.*

Proof. Suppose towards a contradiction that $a_0, \dots a_{n+1}$ are witnesses for T_n being $n+1$-ample over some set of parameters A. We have

$$a_{n+1} \underset{A}{\not\perp} a_0,$$

$$a_{n+1} \underset{Aa_i}{\perp} a_0 \dots a_i, i = 0, \dots n.$$

By the first condition there are vertices in $\operatorname{acl}(a_0)$ and in $\operatorname{acl}(a_{n+1})$ which are in the same connected component. Put $f_0 = \operatorname{proj}(a_{n+1}/a_0 A) \in \operatorname{acl}(a_0 A)$ and $f_{n+1} = \operatorname{proj}(f_0/a_{n+1}A) \in \operatorname{acl}(a_{n+1}A)$.

Since

$$a_{n+1} \underset{Aa_i}{\perp} a_0 \dots a_i, i = 1, \dots n$$

using Corollary 2.19 we inductively find flags

$$f_i = \mathrm{proj}(f_{n+1}/f_0 f_1 \dots f_{i-1} a_i A) = \mathrm{proj}(f_{n+1}/a_i A) \in \mathrm{acl}(a_i A), i = 1, \dots n$$

such that

$$f_{n+1} \underset{f_i}{\downarrow} f_0 f_1 \dots f_i.$$

For $i = 1, \dots n$ we clearly have

$$\mathrm{acl}(f_0, f_1 \dots f_{i-1}, f_i) \cap \mathrm{acl}(f_0, \dots f_{i-1}, f_{i+1}) \subseteq \mathrm{acl}(a_0, \dots a_{i-1} A).$$

By construction, there is a reduced path $\gamma = (f_0, x_1, \dots x_k = f_{n+1})$ containing a vertex of each of the f_i in ascending order. Since we cannot have a flag containing more than n elements, there must be some vertex x in γ where γ changes direction. For some i we then have $x \in f_{i+1}$ or x occurs in γ between an element of f_i and an element of f_{i+1}. By Lemma 2.15 we have $x \in \mathrm{acl}(f_i f_{i+1}) \cap \mathrm{acl}(f_i f_{i+2})$. Then

$$x \underset{f_i}{\downarrow} a_0 a_1, \dots a_i A,$$

so $x \notin \mathrm{acl}(a_0 a_1, \dots a_i A)$, a contradiction. □

The proof shows that in fact the following stronger ampleness result holds:

Corollary 3.3. *If $a_0, \dots a_n$ are witnesses for T_n being n-ample, then there are vertices $b_i \in \mathrm{acl}(a_i)$ such that $(b_0, \dots b_n)$ is a flag.*

4 Buildings and the prime model of T_n

We now turn towards constructing the prime model M_n^0 of T_n as a Hrushovski-limit. We will show that M_n^0 is the building associated to a right-angled Coxeter group.

For this purpose we now consider an expansion L_n' of the language L_n by binary function symbols f_k^i. For an L_n-graph A we put $f_k^i(x, y) = z$ if z is the k^{th} element on a unique shortest E_i-path of length at least k from x to y and $z = x$ otherwise.

We say that an L_n-graph A is E_i-connected if the set $V_{i-1}(A) \cup V_i(A)$ is connected.

Definition 4.1. *Let $\mathcal{K}'$ be the class of finite L_n'-graphs $A \in \mathcal{K}$ which are E_i-connected for $i = 1, \ldots n$ and additionally satisfy the following condition:*

6. *For any $a \in A$ the residue $R(a)$ is E_i-connected for $i = 1, \ldots n$.*

Note that $\mathcal{K}'$ is closed under finitely generated substructures by the choice of language.

Definition 4.2. *Let A be a finite L_n'-graph which is E_i-connected for $i = 1, \ldots n$. The following extensions are called* 1*-point strong extensions of A:*

1. *add a vertex of type V_0 or V_n to A which is connected to at most one vertex of A and such that the extension is still E_i-connected for all $i = 1, \ldots n$.*

2. *If (x, y, z) is a dense flag in A, add a vertex y' of the same type as z to A such that (x, y', z) is a flag.*

3. *if $A \subset V_0 \cup V_n, |A| \leq 1$, add a vertex of appropriate type which is connected to the vertex of A if $A \neq \emptyset$.*

Again we write $A \leq B$ if B arises from A by finitely many 1*-point strong extensions.*

We next show that $(\mathcal{K}_n', \leq)$ has the amalgamation property for $\leq$-extensions.

Lemma 4.3. *If A contains a flag of type $(V_1, \ldots V_{n-1})$ and $A \leq B, C$ are in $\mathcal{K}_n'$, then $D := B \otimes_A C \in \mathcal{K}_n'$ and $B, C \leq D$.*

Proof. Clearly, $B, C \leq D$ and D is E_i-connected for all $i = 1, \ldots n$ since A contains a flag. To see that $D \in \mathcal{K}$, note that if $B \in \mathcal{K}$ and B' is an 1-point strong extension of B, then also $B' \in \mathcal{K}$. □

This shows that the class $(\mathcal{K}, \leq)$ has a Hrushovski limit M_n^0. Clearly, M_n^0 is E_i-connected for $i = 1, \ldots n$ and since any two vertices of M_n^0 lie in a maximal flag, it follows that M_n^0 is in fact n-connected. Note that an L_n'-substructure of M_n^0 is automatically nice, see Remark 2.10.

The same proof as in the case of M_n shows the first part of the following proposition:

Proposition 4.4. *The Hrushovski limit M_n^0 is a model of T_n. Furthermore M_n^0 is the unique countable model of T_n which is E_i-connected for $i = 1, \ldots n$ and such that every vertex is contained in a maximal flag.*

(Note that in [1] the corresponding Remark 3.6 of uses Lemma 3.2, which is not correct as phrased there: M_n^0 and $M_n^0 \cup \{a\}$ with a an isolated point are not isomorphic, but satisfy the assumptions of Remark 3.6.)

The uniqueness part of Proposition 4.4 follows directly from the following theorem and Proposition 5.1 of [4] which states that this type of building is uniquely determined by its associated Coxeter group and the cardinality of the residues.

Theorem 4.1. *M_n^0 is a building of type $A_{\infty,n+1}$ all of whose residues have cardinality $\aleph_0$.*

Recall the following definitions (see e.g. [3]). Let W be the Coxeter group

$$W = \langle t_0, \dots t_n \colon t_i^2 = (t_i t_k)^2 = 1, i, k = 0 \dots n, |k - i| \geq 2 \rangle,$$

whose associated diagram we call $A_{\infty,n+1}$.

Definition 4.5. *A building of type $A_{\infty,n+1}$ is a set Δ with a* Weyl distance function $\delta : \Delta^2 \to W$ *such that the following two axioms hold:*

1. *For each $s \in S := \{t_i, i = 0, \dots n\}$, the relation $x \sim_s y$ defined by $\delta(x, y) \in \{1, s\}$ is an equivalence relation on Δ and each equivalence class of $\sim_s$ has at least 2 elements.*

2. *Let $w = r_1 r_2 \dots r_k$ be a shortest representation of $w \in W$ with $r_i \in S$ and let $x, y \in \Delta$. Then $\delta(x, y) = w$ if and only if there exists a sequence of elements $x, x_0, x_1, \dots, x_k = y$ in Δ with $x_{i-1} \neq x_i$ and $\delta(x_{i-1}, x_i) = r_i$ for $i = 1, \dots, k$.*

A sequence as in 2. is called a *gallery of type* $(r_1, r_2, \dots, r_k)$. The gallery is called reduced if the word $w = r_1 r_2, \dots, r_k$ is reduced, i.e. a shortest representation of w.

We now show how to consider M_n^0 as a building of type $A_{\infty,n+1}$.

Proof. (of Theorem 4.1) We extend the set of edges of the $n + 1$-coloured graph M_n^0 by putting edges between any two vertices that are incident in the sense of Definition 2.11. In this way, flags of M_n^0 correspond to a complete subgraph of this extended graph, which thus forms a simplicial complex. A maximal simplex consists of $n + 1$ vertices each of a different type V_i. (Such a simplex is called a *chamber*.) Let Δ be the set of maximal simplices in this graph. Define $\delta : \Delta^2 \to W$ as follows:

Put $\delta(x,y) = t_i$ if and only if the flags x and y differ exactly in the vertex of type V_i. Extend this by putting $\delta(x,y) = w$ for a reduced word $w = r_1 r_2 \ldots r_k$ if and only if there exists a sequence of elements $x = x_0, x_1, \ldots, x_k = y$ in Δ with $x_{i-1} \neq x_i$ and $\delta(x_{i-1}, x_i) = r_i$ for $i = 1, \ldots, k$.

Clearly, with this definition of δ, the set Δ satisfies the first condition of Definition 4.5. In fact, for all $s \in S$ every equivalence class $\sim_s$ has cardinality $\aleph_0$. We still need to show that δ is well-defined, i.e. we have to show the following for any $x, y \in \Delta$: if there are reduced galleries $x_0 = x, x_1, \ldots, x_k = y$ and $y_0 = x, y_1, \ldots, y_m = y$ of type $(r_1, r_2, \ldots, r_k)$ and $(s_1, \ldots s_m)$, respectively, then in W we have $r_1 r_2 \ldots r_k = s_1 \ldots s_m$. Equivalentely, we will show the following, which completes the proof of Theorem 4.1:

Claim: There is no reduced gallery $a_0, a_1, \ldots, a_k = a_0$ for $k > 0$ in M_n^0.

Proof of Claim. Suppose otherwise. Let $a_0, a_1, \ldots, a_k = a_0$ be a reduced gallery of type $(r_1, \ldots r_k)$ for some $k > 0$. Note that the flags a_i and a_{i+1} contain the same vertex of type V_j as long as $r_i \neq t_j$.

Now consider the sequence of vertices of type V_n and V_{n-1} occurring in this gallery. Since $V_n \cup V_{n-1}$ contains no cycles, the sequence of vertices of type V_n and V_{n-1} occurring in this gallery will be of the form

$$(x_1, y_1, x_2, y_2, \ldots, x_i, y_i, x_i, y_{i-1}, \ldots y_1, x_1) \tag{1}$$

with $x_i \in V_n, y_i \in V_{n-1}$ and x_i a neighbour of y_i and y_{i-1} in the original graph. This implies that at some place in the gallery type there are two occurrences of t_n which are not separated by an occurrence of t_{n-1} (or conversely). Since t_n commutes with all t_i for $i \neq n-1$ and the word $r_1 \ldots r_k$ is reduced, there are two occurrences of t_{n-1} which are not separated by an occurrence of t_n, say $r_j, r_{j+m} = t_{n-1}$ with $r_{j+1}, \ldots, r_{j+m-1} \neq t_n$.

We now consider the gallery $a_j, \ldots a_{j+m}$ of type $(r_j = t_{n-1}, r_{j+1}, \ldots, r_{j+m} = t_{n-1})$. Notice that by (1), the flags a_j and a_{j+m} have the same V_n and the same V_{n-1} vertex. Since M_n^0 does not contain any E_{n-1}-cycles, the sequence of V_{n-1}- and V_{n-2}-vertices appearing in this sequence must again be of the same form as in (1). Exactly as before we find two occurrences[1] of t_{n-2} in the gallery type of $a_j, \ldots a_{j+m}$ which are not separated by an occurrence of t_{n-1}. Continuiung in this way, we eventually find two occurrences of t_1 which are not separated by any t_i. Since $t_1^2 = 1$ this contradicts the assumption that the gallery be reduced. ∎

[1]If t_{n-2} does not occur in the type of the gallery, this would contradict the assumption that the type is reduced since t_{n-1} commutes with all t_i for $i \neq n, n-2$.

The proof shows in fact the following:

Corollary 4.6. *A model of T_n is a building of type $A_{\infty,n+1}$ if and only if it is E_i-connected for all i and every vertex is contained in a maximal flag.*

Theorem 4.2. *The building M_n^0 is the prime model of T_n*

Proof. To see that M_n^0 is the prime model of T_n note that for any flags $C_1, C_2 \in M_n^0$ and gallery $C_1, = x_0, \ldots, x_k = C_2$ the set of vertices occuring in this gallery is E_i-connected for all i. Hence by Remark 2.10 its type is determined by the quantifier-free type.

Thus, given a maximal flag M in any model of T_n and a maximal flag c_0 of M_n^0 we can embed M_n^0 into M by moving along the galleries of M_n^0. ☐

5 Ranks and types

Recall that for vertices $x, y \in M_n^0$ with $x \in V_i, y \in V_j$ the *Weyl-distance* $\delta(x, y)$ equals $w \in W$ if there are flags C_1, C_2 containing x, y, respectively, with $\delta(C_1, C_2) = w'$ and such that w is the shortest representative of the double coset $\langle t_k \colon k \neq i\rangle^W w' \langle t_k \colon k \neq j\rangle^W$ (where as usual $\langle X\rangle^W$ denotes the subgroup of W generated by X).

The following is clear:

Proposition 5.1. *The theory T_n has quantifier elimination in a language containing predicates $\delta_w^{i,j}$ for Weyl distances between vertices of type V_i and of type V_j.*

For any small set A in a large saturated model we have the following kinds of regular types:

(I) $\mathrm{tp}(a/A)$ where $a \in V_i$ is not connected to any element in $\mathrm{acl}(A)$

(II) $\mathrm{tp}(a/A)$ where $a \in V_i$ is incident with some $b \in \mathrm{acl}(A) \cap V_j$ but not connected in $R(b)$ to any vertex in $\mathrm{acl}(A) \cap R(b)$.

(III) $\mathrm{tp}(a/A)$ where $a \in V_i$ is incident with some $x, y \in \mathrm{acl}(A)$ such that (x, a, y) is a flag with $x \in V_k, y \in V_j$; and as a special case of this we have

(IV) $\mathrm{tp}(a/A)$ where $a \in V_i$ has neighbours $x, y \in \mathrm{acl}(A)$ such that (x, a, y) is a (necessarily dense) flag.

By quantifier elimination any of these descriptions determines a complete type. Using the description of forking in Corollary 2.19 one sees easily that each of these types is regular and trivial.

Clearly, any type in (IV) has U-rank 1 and in fact Morley rank 1 by quantifier elimination. It also follows easily that $\mathrm{MR}(a/A) = \omega^n$ if $\mathrm{tp}(a/A)$ is as in (I). In case (II) we find that $\mathrm{MR}(a/A) = \omega^{n-j-1}$ or $\mathrm{MR}(a/A) = \omega^{j-1}$ depending on whether or not $i < j$. In case (II) we have $\mathrm{MR}(a/A) = \omega^{|k-j|-2}$.

Just as in [1] we obtain:

Lemma 5.2. *Any regular type in T_n is non-orthogonal to a type as in (I), (II), or (III).*

Proof. Let $p = \mathrm{tp}(b/\operatorname{acl}(B))$. If b is not connected to $\operatorname{acl}(B)$, then p is as in (I), so we may assume that $\mathrm{proj}(b/B) = C \neq \emptyset$. Let a be a vertex on a short path from b to C incident with an element of C. Then by Corollary 2.19 we see that p is non-orthogonal to $\mathrm{tp}(a/C)$ and $\mathrm{tp}(a/C)$ is of type (II) or (III). $\square$

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